From solar cells to ocean buoys: Wide-bandwidth limits to absorption by metaparticle arrays

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In this paper, we develop an approximate wide-bandwidth upper bound to the absorption enhancement in arrays of metaparticles, applicable to general wave-scattering problems and motivated here by ocean-buoy energy extraction. We show that general limits, including the well-known Yablonovitch result in solar cells, arise from reciprocity conditions. The use of reciprocity in the stochastic regime leads us to a diffusion model from which we derive our main result: an analytical prediction of optimal array absorption that closely matches exact simulations for both random and optimized arrays. This result also enables us to propose and quantify approaches to increase performance through careful particle design and/or using external reflectors.

Introduction. One of the most influential theoretical results for solar-cell design has been the Yablonovitch limit [1–8], which provides an approximate bound to how much surface texturing can enhance the performance of an absorbing film averaged over a broad bandwidth and angular range, and which depends only on the refractive index of the film. In this paper, we apply a similar philosophy, but very different mathematical techniques, to obtain approximate broad-band/angle absorption limits for a case in which the traditional Yablonovitch result is not useful: dilute arrays of “metaparticles” (synthetic absorbers/scatterers). This problem arises in optics contexts [9–11], but we were actually motivated by arrays of buoys designed to extract energy from ocean waves [12, 13] depicted in Fig. 1, in particular by a recent extensive computational study on large arrays [14], and our analysis works for any kind of wave equation. We suppose that the absorbing/scattering properties of an individual metaparticle (buoy) are known, and the question is how much enhancement can be gained by placing the buoys near one another in an array, via multiple-scattering effects. Previous numerical-optimization work [14–18], showed that designing the particle positions could yield substantial gains, but in this paper we derive a more general result that is independent of the particle arrangement. With no free parameters, using only the individual-particle properties and the array density, we obtain a simple radiative-diffusion model (Eq. 2) that predicts with better than 5% accuracy the angle/frequency-averaged performance of the optimized array, and also the frequency-averaged performance of random arrays. This model allows us to quickly evaluate the performance benefits of different metaparticle designs and array configurations, and we show that substantial improvements are possible if the scattering cross-section is increased (relative to the absorption cross-section) and/or if partially reflecting strips are placed on either side of the array. Like the Yablonovitch limit, we believe that our model sets a fundamental benchmark that will guide the development of a wide variety of scattering/absorbing-particle applications, in ocean-power generation and elsewhere.

The key reason why the Yablonovitch limit cannot be applied to all metaparticle arrays is that it requires an effective-medium (homogenization) approximation (i.e., an effective index), which is only accurate for either dilute weakly interacting dipolar particles [19] or for strongly interacting particles with sufficiently subwavelength separation [20], neither of which is true of the ocean-power problem. Moreover, the Yablonovitch limit is independent of the precise nature of the scattering texture, whereas in our case the whole point is to extrapolate the array properties from the individual-scatterer properties. (These properties are complicated for ocean buoys, because in addition to the “hard” scattering process there is a radiative process due to induced buoy motion, with both propagating and evanescent channels [14, 21, 22].) Nevertheless, the spirit of our approach is closely tied to that of the Yablonovitch limit, which was originally based on the realization that isotropic light propagation is ideal

![Figure 1: Left: We bound absorption for very general arrays of “particles”, including arrays of buoys that extract energy from ocean waves [12, 13]. Right: Ocean surface displacement for a cylindrical buoy array [14] with incident waves from left (arrow).](image-url)
for broad-band absorbers in the geometric-optics limit [1, 2]. In both the solar-cell case (reviewed in the SM) and in our system, we derive this isotropic property from general wave-reciprocity arguments. In our case, this approach leads us within a stochastic approximation to a radiative-diffusion equation that accurately describes the absorption averaged over a broad bandwidth and range of incidence angles (Fig. 2). Although our equations are more complicated than the simple $4\pi^2$ Yablonovitch factor, they allow one to do a simple calculation once for a given absorbing region and particle density, and apply the result to any arrangement of particles, without ever solving a multiple-scattering wave problem.

**Problem.** We consider a two-dimensional array of scattering/absorbing particles distributed inside a layer along the $y$-axis (e.g. Fig. 1). Our first goal is to predict, for a given incident angle $\theta$, the interaction factor $q(\theta)$ [15, 23], the ratio of the power extracted by the array to that of the equivalent number of isolated particles. This factor is often smaller than 1 for strongly absorbing particles because the incident wave is attenuated before reaching the rear of the multi-row array.

**Reciprocity.** The original intuition behind the ray-optical Yablonovitch limit is that the optimal enhancement is achieved through an isotropic distribution of light inside the device [1, 2]. This can be thought of as a reciprocity condition. Reciprocity [23] implies that rays at a given position cannot emerge in the same direction from two different paths. In consequence, if a given point in the absorber is to be reached from as many ray bounces as possible, the rays must be entering/exiting that point from all angles. More formally, we show in SM2 that reciprocity can be applied to the full Maxwell’s equations in order to relate the enhancement to the density of states (accomplished in another way by Ref. [25], again recovering the Yablonovitch limit in bulk media.

A similar procedure can be followed in the ocean-buoy problem. By applying the appropriate reciprocity relation derived from the wave equation, the Haskind–Hanaoka formula [24], to the maximum absorption of an array of buoys [23], one finds a rigorous limit on the total absorption cross section of the array [26, SM3]. The result implies that for isotropic incidence, the interaction factor is at most 1 at the resonance frequency (the frequency at which the single buoy reaches its maximum absorption). In this Letter, we generalize this result to non-optimal buoys for any distribution of incident waves. Then, we derive a tighter bound that predicts the frequency-averaged performance of random arrays and the angle/frequency-averaged performance of optimized arrays.

**General limit.** For a sufficiently dilute array, one can neglect near-field interactions between scatterers. And for either a random array or for a sufficiently wide band-angle average, coherent scattering effects average out. These two conditions allow one to write an equation only involving specific intensity $I$: the radiative transfer equation (RTE) [24, 27]. The interaction factor can then be estimated as $\langle f_0^{2\pi} I(r, \theta')d\theta'\rangle_r/I_r$ where $I_r$ is the incident intensity.

For this problem, one can define a surface Green’s function $G_s(r, \theta'; \theta)$ [29] giving $I(r, \theta')$ for a field incident from angle $\theta$. From reciprocity [30, SM4], it follows that $\int_{\Delta_s} G_s(r, \theta'; \theta)d\theta \leq 1$, with equality reached for negligible absorption. We can therefore bound the $q$ for a given directional spectrum $f(\theta)$ [fraction of power incident from angle $\theta$]:

$$
\langle q \rangle = \int_{2\pi} \int_{2\pi} f(\theta)(G_s(r, \theta'; \theta))_r d\theta' d\theta \leq 2\pi \max_{\theta} f
$$

with equality reached for $G_s(r, \theta'; \theta) = \delta(\theta - \theta_m)$ where $\theta_m = \arg \max f$. Since [1] does not assume an optimal single-buoy absorption, it generalizes the previous bound [20], giving $\langle q \rangle \leq 1$ for isotropic incidence $f = 1/2\pi$ at any wavelength.

A key result from [1] is that the maximum $q$ is attained for a $G_s$ independent of $\theta'$, corresponding to isotropic interior intensity, similar to the Yablonovitch model. Therefore, in order to understand whether the upper bound [1] can be reached through the particles’
scattering, we solve the RTE under the assumption of nearly isotropic intensity, which is well known to lead to a diffusion model [24, 27, 29, 31].

Radiative-diffusion model. The key properties for the radiative-diffusion model are the scattering/absorption cross sections ($\sigma_s$, $\sigma_a$) [24, 27] and the asymmetry factor ($\mu$) [32] of a single particle (Fig. 3), the linear density of the array along the $y$-axis ($n_t$), and the reflection coefficients at the boundaries ($R_t$). By defining the cross sections per unit length as $(v_s, v_a) = n_t(\sigma_s, \sigma_a)$, our model predicts a $q$ of:

$$q(\theta) = q_0(\theta) \left( \eta \left[ D \frac{\xi(v_d)}{\xi(v_e \sec \theta)} + C \right] + 1 \right) \quad (2)$$

where $v_c = v_a + v_s$ and $v_c^2 = 2v_a(v_c - v_s\mu)$ are the extinction and diffusion coefficients, $C = [2v_s(v_c + \mu v_a)]/[v_c^2 - (v_c \sec \theta)^2]$, $\xi(x)$ is the function $(1 - e^{-x})/x$, and $q_0$ is the single-scattering factor. General formulas for $q_0(\theta)$, $D$, and $\eta$ are given in SM1, but in the absence of reflecting walls ($R_t = 0$) they simplify to $q_0(\theta) = \xi(v_e \sec \theta)$ and:

$$D = \frac{C(1 + e^{-v_c \sec \theta}) + \frac{\pi(C + 2p_1 \cos^2 \theta)}{4(1 - p_1 \cos \theta)}(1 - e^{-v_c \sec \theta})}{(1 + e^{-v_a}) + \frac{\pi v_d}{4v_c(1 - p_1)}(1 - e^{-v_d})}, \quad (3)$$

$$\eta = \frac{\pi}{2} - \frac{\pi}{2} - \frac{4}{v_c} - \frac{4}{v_c} e_{v_e}^{(1)} - \frac{4}{v_c} e_{v_e}^{(2)} - \frac{2}{v_c} e_{v_e}^{(3)} \quad (4)$$

where $p_1 = \sigma_\mu / \sigma_e$ and $e_{\mu}^{(i)} = \int_0^{\pi/2} e^{-x \sec \alpha} \cos \alpha \, d\alpha$.

Equation (2) with $\eta = 1$ (standard diffusion model) is obtained by solving the radiative-diffusion equations with flux-matching boundary conditions [24, 27] as reviewed in SM4. However, it is also known that the diffusion solution is inaccurate for small thicknesses [33, 35]. One can apply a correction [34] based on the fact that $q = 1$ in the RTE for isotropic incidence and negligible absorption. Here, this correction can be achieved with the scalar multiplicative factor $\eta$ given in Eq. 4 [SM4] because we only need the total $q$ and not the spatially resolved $\eta$; $\eta \to 1$ for an absorber that is thick compared to the extinction length.

Ocean-buoy arrays. We now present a validation of the accuracy of (1) in a model ocean-buoy system with a truncated cylinder (Fig. 1). The isolated-buoy properties can be obtained analytically [36, 38] and are depicted in Figure 3: they are designed [14] to have an absorption resonance that matches the typical Bretschneider spectrum [39] of ocean waves. We choose the array density based on an earlier optimized periodic 3-row buoy arrangement [14]. For this density, we then compare the exact numerical scattering solution calculated for both random and optimized-periodic arrays (using the method from [14]) to both the analytical radiation-diffusion $q$ from (2), with and without the correction $\eta$, and the numerical solution of the RTE model by a Monte Carlo method [10].

In Fig. 2, our corrected model agrees to $< 2\%$ accuracy with exact solutions for random arrays at $\theta < 80^\circ$, as long as the results are frequency-averaged. The importance of frequency averaging is shown by the $q$ frequency spectrum shown in the inset for $\theta = 0^\circ$. For an ensemble of random structures, this spectrum exhibits a large standard deviation (gray shaded region), due to the many resonance peaks that are typical of absorption by randomized thin films [3, 5], but the frequency average mostly eliminates this variance and matches our predicted $q(\theta)$. Precisely such an average over many resonances is what allows the Yablonovitch model to accurately predict the performance of textured solar cells even though it cannot reproduce the detailed spectrum [3, 41].

At first glance, our model does not agree in Fig. 2 with the performance of the optimized periodic array from Ref. [14] the periodic array, which was optimized for waves near normal incidence, is better at $\theta$ near $0^\circ$ and worse elsewhere. However, when we also average over $\theta$ (from a typical ocean-wave directional spectrum [28]), the result (shown as a parenthesized number in the legend of Fig. 2) matches (2) within 5%. If we average over all angles assuming an isotropic distribution of incident waves, the results match within 1%. Similar results have been observed for thin-film solar cells, in which an opti-
tering cross section on the bandwidth-averaged factor \( q_s \) for the same array in Fig. 2. We tune the index \( n_1 \) along a strip surrounding the array, with \( n_0 \) being the index of the array’s ambient medium. We suppose that the buoy has new scattering cross section \( \sigma_s \), but keep the same absorption cross section. Left: \( q_s \) at normal incidence. Right: \( q_s \) averaged over \( \theta \) with a directional spectrum of \( \cos^2 \theta \) and \( s = 4 \).

Larger interaction factor. Given this model, we can now explore ways to increase \( q \). By examining the dependence of \( q \) in (2) on the parameters, we find that for a fixed absorption-to-scattering ratio \( \sigma_a/\sigma_s \), \( q \) reaches a maximum \( q_{\text{max}} \) for an optimal value of scattering per unit length \( \sigma_s n_1 \), whereas it increases monotonically with \( \mu \) (forward scattering). The optimal value of \( \sigma_s n_1 \) and \( q_{\text{max}} \) both decrease with \( \sigma_a/\sigma_s \); as the single particle absorbs more, the interaction factor decreases and the optimal configuration requires a larger spacing between the particles. In essence, more scattering is typically better, as quantified in Fig. 4.

From Fig. 3, we see that we have \( \sigma_a/\sigma_s \approx 1 \) at the resonance of the ocean buoy. In this case, the enhancement is expected to be smaller than 1 around the resonance and the optimal structures will tend to have a large spacing \( d_g \). If the array were optimized for small wavelengths \( \lambda \), where \( \sigma_s \gg \sigma_a \), then a larger \( q \) could be obtained at those wavelengths, but overall performance would be worse because the optimal spacing in this case is too small for good performance at the resonance.

Alternatively, we show that \( q \) can be enhanced by putting partially reflecting strips around the array. Similar to light-trapping by total internal reflection [1, 2], one possibility is to use a strip of a lower-“index” medium (compared to the array’s ambient medium) on either side of the array. In the ocean-buoy problem, this can for example be achieved by either a depth change or the use of a tension/bending surface membrane which can lead to near-zero index [42, 43]. This modifies (2–4) by the effect of reflection coefficients \( R \), as given in SM1.

In Fig. 4, we show the effect of an increase in the scattering cross section and/or the index contrast for the same array studied before. By combining both effects, a large (> 3) spectral interaction factor can be achieved at normal incidence. At the same time, waves incident at large angles will be reflected out, so that the interaction factor integrated over isotropic incidence is still smaller than 1. For a given directional spectrum and scattering cross section of a single buoy, the optimal interaction factor is achieved for a specific value of the index contrast as can be seen in Fig. 4 (right).

Finally, it is instructive to look at the ideal case of small absorption \( (n_1 \sigma_a \ll 1) \) and large scattering \( (n_1 \sigma_s \gg 1) \), for which the result simplifies to:

\[
q(\theta) = \left[ 1 - R(\theta) \right] \left[ \frac{\pi}{4\alpha} + \cos \theta \right] \cos \theta
\]

where \( R \) is the reflection coefficient of the surface and \( \alpha = (1 - r_1)/(1 + r_2) \) with \( r_1 = \int_{-\pi/2}^{\pi/2} R(\theta) \cos^2(\theta) \cos \theta \). Equation 5 still gives 1 when averaged over isotropic incidence, but the interaction factor at normal incidence is larger. Without reflectors \( (R = 0) \), the maximum value of \( q \) at normal incidence is \( 1 + \frac{\pi}{2} \). Although this is a maximum limit of the large-bandwidth \( q(0) \) in this regime without reflectors, \( q(0) \) can still be made arbitrarily large by including a reflector designed for transmission near normal incidence and reflection elsewhere.

Conclusion. We believe that the angle/frequency-averaged limits presented in this paper provide guidelines for future designs to achieve a large \( q \) factor which may open the path for the realization of large arrays of buoys for efficient ocean energy harvesting. The results are also applicable to other problems where multiple scattering effects are used to achieve enhancement, including scattering particles inside an absorbing layer. For three-dimensional arrays, the coefficients in (2) are slightly modified [SM5]. One can recover the standard \( 4n^2 \) result from our approach in an appropriate limit [SM5], but the real power of our result is that it works for new regimes of strongly absorbing particles/medium and angle-resolved bounds.

Supplemental Material

From solar cells to ocean buoys:
Wide-bandwidth limits to absorption by metaparticle arrays

1. COMPLETE EXPRESSION FOR THE INTERACTION FACTOR

In this section, we give the complete expression for \( q \) in the presence of reflecting boundaries. The derivation is given in section 4.

The key properties for the diffusion-radiative model are the scattering/absorption cross sections \( (\sigma_s, \sigma_a) \) [1, 2] and the asymmetry factor \( (\mu) \) [3] of a single particle (Fig. 3 of main text), the linear density of the array along the \( y \)-axis \( (n_t) \), and the reflection coefficients at the boundaries \( (R_t) \). This allows us to define the cross sections per unit of length \( (v_s, v_a) = n_t(\sigma_s, \sigma_a) \). Given these properties and the incident angle \( \theta \), the radiative-diffusion model (below) predicts a \( q \) of:

\[
q(\theta) = q_0(\theta) \left( \frac{D \xi(v_d)}{\xi(v_c \sec \theta)} + C \right) + 1
\]  

(1.1)

where \( v_c = v_a + v_s \) and \( v^2_s = 2v_s(v_c - v_s \mu) \) are the extinction and diffusion coefficients, \( C = 2[v_s(v_c + \mu v_a)]/[v^2_d - (v_c \sec \theta)^2] \), and \( \xi(x) \) is the function \((1 - e^{-x})/x\).

\( q_0(\theta) \) is the single-scattering factor given by:

\[
q_0(\theta) = \frac{(1 - \hat{R}_1)(1 + \hat{R}_2 Y)}{1 - \hat{R}_1 \hat{R}_2 Y^2} \xi(v_c \sec \theta)
\]  

(1.2)

with \( \hat{R}_1 = R_i(\theta) \) and \( Y = e^{-v_c \sec \theta} \).

\( D \) is given through boundary conditions by \( D = \frac{A + B}{1 + \hat{R}_2 Y} \), where:

\[
\begin{bmatrix}
\alpha_1 + \frac{\pi v_d}{4 v_r} e^{-v_d} & (\alpha_1 - \frac{\pi v_d}{4 v_r}) e^{-v_d} \\
(\alpha_2 - \frac{\pi v_d}{4 v_r}) e^{-v_d} & (\alpha_2 + \frac{\pi v_d}{4 v_r}) e^{-v_d}
\end{bmatrix}
\begin{bmatrix}
A \\
B
\end{bmatrix}
= -\frac{C(1 + \hat{R}_2 Y^2) \alpha_1 + \frac{\pi v_s}{4 v_r} (\frac{\sigma_a}{\cos \theta} + 2p_1 \cos \theta)(1 - \hat{R}_2 Y^2)}{C(1 + \hat{R}_2) \alpha_2 - \frac{\pi v_s}{4 v_r} (\frac{\sigma_a}{\cos \theta} + 2p_1 \cos \theta)(1 - \hat{R}_2))Y}
\]  

(1.3)

with \( \alpha_i = (1 - r_i^2)/(1 + r_i^2), r^2_p = \int_{-\pi/2}^{\pi/2} R(\theta) \cos^2(\theta) d\theta/\int_{-\pi/2}^{\pi/2} \cos^2(\theta) d\theta, v_c p_1 = v_s \mu \) and \( v_tr = v_c(1 - p_1) \).

\( \eta \) is a correction term to the standard diffusion solution to ensure that the interaction factor for isotropic incidence and zero absorption is 1. \( \eta \) is defined as:

\[
\eta = \frac{\pi - \int_{0}^{\pi/2} [q_0^{(1)}(\theta, v_c, v_tr) + q_0^{(2)}(\theta, v_c, v_tr)]/\xi(v_c \sec \theta) - 2 \cos^2 \theta(q_0^{(1)} + q_0^{(2)}) d\theta}{\int_{0}^{\pi/2} [q_0^{(1)}(\theta, v_c, v_tr) + q_0^{(2)}(\theta, v_c, v_tr)]/\xi(v_c \sec \theta) - 2 \cos^2 \theta(q_0^{(1)} + q_0^{(2)}) d\theta}
\]  

(1.4)

with:

\[
(1 + \hat{R}_2 Y) D_0(\theta, v_c, v_tr) = \frac{(\alpha_2 + \frac{\pi v_s}{4 v_r} \cos \theta)(X_1 + (\alpha_1 + \frac{\pi v_s}{4 v_r} \cos \theta)(X_2) + 2 \alpha_1 \alpha_2}{X}
\]  

(1.5)

where:

\[
X = \begin{bmatrix}
2 \cos^2 \theta(1 + \hat{R}_2 Y^2) \alpha_1 + \frac{\pi}{2} \cos \theta(1 - \hat{R}_2 Y^2) \\
2 \cos^2 \theta(1 + \hat{R}_2 Y^2) \alpha_2 - \frac{\pi}{2} \cos \theta(1 - \hat{R}_2 Y^2)
\end{bmatrix}
\]  

(1.6)

Superscripts for \( q_0^{(i)} \) and \( D_0^{(i)} \) refer to the boundary that is facing the incident wave.

2. ENHANCEMENT FROM RECIPROCITY OF MAXWELL’S EQUATIONS

Although the end result is not new, we wish to emphasize that the underlying ideas of the Yablonovitch and LDOS limits are closely tied to reciprocity, which we use in a different way in our metaparticle result. This is an alternative
to the derivation in [4], which differs in that it directly uses the reciprocity (or generalized reciprocity) from Maxwell’s equations. As was also emphasized in [4], the result also applies to linear nonreciprocal systems, since the density of states of transposed-related materials is the same ($G(r, r) = G^T(r, r)$ [2]).

Here for simplicity, we consider a reciprocal system in the derivation. We have then:

$$\int_{S, \omega} [\mathbf{E}_a \times \mathbf{H}_b - \mathbf{E}_b \times \mathbf{H}_a] \cdot \hat{k} \, dS = \int_V (\mathbf{E}_a \cdot \mathbf{J}_b - \mathbf{E}_b \cdot \mathbf{J}_a) \, dV \tag{2.1}$$

If we choose $\mathbf{J}_a = \frac{1}{j\mu} \hat{e}_a \delta_r \mathbf{r}_0$ and $\mathbf{E}_b^{\text{inc}} = 0$ (and $\mathbf{J}_b = 0$, $\mathbf{E}_b^{\text{inc}} = e^{jk_0 \cdot \mathbf{r}_0} \hat{e}_b$), then $\mathbf{E}_a = \mathbf{G}_E(r_0, \mathbf{r}_0) \hat{e}_a$.

The field term can be written as: $\mathbf{E}_a^* = f_s(\hat{k}) e^{jk_0 \cdot \mathbf{r}_0} \hat{e}_a$, $\mathbf{B}_a = \frac{1}{(\hat{k} \times \mathbf{E}_a^*)}$ with $\eta = \frac{\sqrt{\epsilon_0}}{\sqrt{\mu}}$, and similarly for the far-field of the scattered field $\mathbf{b}$, so that: $\int_{S, \omega} [\mathbf{E}_a^* \times \mathbf{H}_b - \mathbf{E}_b^* \times \mathbf{H}_a] \cdot \hat{k} \, dS = 0$.

We then expand the integrand of the left term in 2.1 to obtain:

$$\int_{S, \omega} [\mathbf{E}_a^* \times \mathbf{H}_b^{\text{inc}} - \mathbf{E}_b^* \times \mathbf{H}_a^*] = -\frac{1}{\eta} \int f_s(\hat{k}) e^{jk_0 \cdot \mathbf{r}_0} ((\hat{e}_a \cdot \hat{e}_b)(1 - \hat{k} \cdot \hat{k}_0) + (\hat{e}_a \cdot \hat{k}_0)(\hat{e}_b \cdot \hat{k})) \, d\hat{k} \tag{2.2}$$

The integral can be evaluated using the method of stationary phase [5]. The function $g(\theta, \phi) = 1 + \hat{k} \cdot \hat{k}_0 = 1 + \cos \theta \cos \theta_0 + \sin \theta \sin \theta_0 \cos(\phi - \phi_0)$ has two extrema at $\pm \hat{k}_0$. The integrand is null at the first, so only the second matters. The Hessian matrix at $-\hat{k}_0$ is given by: $\left[ \begin{array}{ll} 1 & 0 \\ 0 & \sin^2 \theta_0 \end{array} \right]$. We then conclude that the integral we want to evaluate is equal to:

$$\int_{S, \omega} [\mathbf{E}_a^* \times \mathbf{H}_b^{\text{inc}} - \mathbf{E}_b^* \times \mathbf{H}_a^*] = -\frac{1}{\eta} \int f_s(\hat{k}) e^{jk_0 \cdot \mathbf{r}_0} ((\hat{e}_a \cdot \hat{e}_b)(1 - \hat{k} \cdot \hat{k}_0) + (\hat{e}_a \cdot \hat{k}_0)(\hat{e}_b \cdot \hat{k})) \, d\hat{k} \tag{2.3}$$

where $\hat{e}_a$ is evaluated at $-\hat{k}_0$.

We finally conclude from 2.1 that:

$$- \hat{e}_a \cdot \mathbf{E}_b(r_0) = 4\pi (\hat{e}_a \cdot \hat{e}_b) f_s(-\hat{k}_0) \tag{2.4}$$

which is the reciprocity relation relating the far field of a point source at $r_0$ in the direction $-\hat{k}_0$ to the field at $r_0$ due to an incoming plane wave from the same direction.

Now, we use the Poynting theorem to compute the far field of the point source:

$$\frac{1}{\eta} \int |f_s(\hat{k})|^2 \, d\hat{k} = \int \mathbf{E}_a \times \mathbf{H}_a^* \cdot \hat{k} \, dS \leq - \int \mathbf{E}_a \cdot \mathbf{J}_a = Im(\mathbf{E}_a \cdot \hat{e}_a) \frac{1}{\omega \mu} \tag{2.5}$$

At this point we are able to combine 2.4 and 2.5 to find our main result about the enhancement. By integrating over all coming angles and polarizations of the “b” field, we have:

$$\int_{\hat{e}_b} |\mathbf{E}_b|^2 \, d\hat{k}_0 = \int_{\hat{e}_b, \hat{e}_a} |\mathbf{E}_b \cdot \hat{e}_a|^2 \, d\hat{k}_0 = (4\pi)^2 \int \sum_{\hat{e}_a, \hat{e}_b} |\hat{e}_a \cdot \hat{e}_b|^2 |f_s(-\hat{k}_0)|^2 \, d\hat{k}_0 \tag{2.6}$$

which relates rigorously the enhancement and the local density of states.

We can use this result to compute the absorbed power and deduce the enhancement compared to the single pass for a cell of surface $S$ and effective thickness $d$. We have:

$$P_{\text{abs}} = \frac{1}{2} \epsilon'' \epsilon_0 \int_V \sum_{\hat{e}_b} |\mathbf{E}_b|^2 \, d\hat{k}_0 \leq \frac{1}{2} \epsilon'' \epsilon_0 (4\pi)^2 \frac{\pi c^3}{2\omega n^2} \int_V \rho \tag{2.7}$$

The incident power for the isotropic incidence and the two polarizations is $\frac{1}{4\pi} \int |\cos \theta| \, d\Omega \times 2 \times S = \frac{2\pi S}{\eta}$ and the normalized single pass absorption is $\alpha d = \epsilon'' \frac{\omega c}{n} d$. The enhancement is then given by:

$$E = \frac{P_{\text{abs}}}{P_{\text{inc}} \alpha d} \leq \frac{2}{n} \frac{\rho}{\eta} \tag{2.8}$$
where \( \rho_v = \frac{\omega^2}{\pi^2c^3} \) is the free space density of states. This inequality becomes indeed an equality in the case of negligible absorption.

For a bulk dielectric, we have: \( \rho = n^3 \rho_v \) so that \( q = 2n^2 \) which is the standard limit in the absence of a back reflector for isotropic incident light.

For an incoming angular distribution \( f(\theta) \) with a normalized flux \( (\int_{4\pi} |\cos \theta| f(\theta) d\Omega = \int_{4\pi} |\cos \theta| d\Omega = 2\pi) \), we then have to multiply the integrand of the first term in (2.6) by \( f(\theta) \) which leads to:

\[
E \leq \frac{2}{n} \rho_v \max f 
\]

(2.9)

To reach this limit, the field should be null for any incident angle different than the one corresponding to the maximum incident amplitude. This immediately gives the factor of 2 for light incident from only a half-space which can be achieved using a back reflector.

We also recover the special case of an isotropic incidence within a cone defined by \( \theta_i \) for a bulk medium (\( f = \frac{2}{\sin \theta_i} \delta(\theta < \theta_i) \)): \( E = \frac{4\pi^2}{\sin \theta_i^2} \).

We finally mention that (2.9) becomes an equality for isotropic incidence and negligible absorption.

### 3. Interaction Factor from Reciprocity in Ocean Waves

In this section, we review a straightforward generalization of the result in Ref. 6 for the case of a general angular distribution. The result is also a consequence of reciprocity, which shows the similarity with the LDOS limit in solar cells.

The problem of ocean wave energy extraction using oscillating bodies is formally equivalent to the problem where there are discrete sources of which the amplitude can in principle be controlled externally (velocity of the body that can be controlled through an external mechanical mechanism). Considering the effect of the incoming wave and interaction between bodies, the total absorption can be written as a quadratic function in terms of the amplitudes of the different sources as in [7] for example. Maximizing the absorption allows to find the optimal amplitudes as a function of the scattered field and the radiated fields from the sources. This gives [7]:

\[
P_{\text{max}} = \frac{1}{8} F_e^*(\theta) R^{-1} F_e(\theta) 
\]

(3.1)

where \( F_e(\theta) \) is the force applied on the bodies for an incident wave from the direction \( \theta \) and \( R \) is the resistance matrix (radiation damping matrix).

One would try to see the effect of the reciprocity relations discussed before on the maximum absorption in this context. The exact equivalent of equation 2.4 is already known in the ocean waves problem as the Haskind-Hanaoka formula that relates the force applied on a body due to an incident wave to the radiated field when the the body acts as a source [8]. It leads to:

\[
F_{e,i}(\theta) = -\frac{4}{k} \rho g A c_p A_i (\theta + \pi) 
\]

(3.2)

where \( A \) is the amplitude of the incident wave, \( A_i \) is the far-field amplitude of the radiation mode \( i \), \( k \) is the wavenumber, \( c_p \) is the group velocity, \( \rho \) is the water density, and \( g \) is the gravity of Earth.

The use of this formula on the maximum absorbed power by an array of oscillating bodies leads to the bound on the power absorbed by the array. For a given incident angular distribution \( f(\theta) \) normalized so that \( \int_{2\pi} f(\theta) d\theta = 1 \):

\[
\langle P_{\text{max}} \rangle = \int f(\theta) P_{\text{max}}(\theta) d\theta \leq \max \frac{1}{\theta} \int P_{\text{max}}(\theta) d\theta = \max \frac{1}{\theta} \sum_{i,j} R_{i,j}^{-1} \int_{2\pi} F_{e,i}^* F_{e,j} d\theta 
\]

(3.3)

Using 3.2 and the fact that \( R_{i,j} = \frac{2}{\pi k} \rho g c_p \) \( \text{Re}(f(2\pi A_i^* A_j)) \) [7], we conclude that:

\[
\langle \sigma_{a,\text{max}} \rangle \leq \frac{NM}{k} 2\pi \max f 
\]

(3.4)

where \( \sigma_{a,\text{max}}^N \) is the maximum absorption cross section of the array, \( N \) is the number of buoys, and \( M \) is the number of degrees of freedom for the buoy motion (1–6 [7], e.g. 1 for only heave motion). This result is general and does not depend on assumptions on the scatterers. It means that the interaction factor \( \sigma_{a,\text{max}}^N / N \sigma_{a}^N \) is bounded by 1 for isotropic incidence [6]. However, it is important to realize that this only applies at the resonance frequency [the \( k \) where the denominator \( \sigma_{a}^N \) reaches the maximum (3.4)].
4. RADIATIVE-DIFFUSION MODEL

In this section, we review the derivation of the radiative-diffusion result in the main text.

We consider a medium containing a distribution of random particles with a scattering cross section $\sigma_s$, an absorption cross section $\sigma_a$, a normalized differential cross section $p(\hat{s}, \hat{s}')$ and a density $n_0$ [2]. If the distance between the particles is large enough so that we can neglect the near field and if the particle separation is random enough so that we can neglect interferences\(^1\), the total differential cross sections for a set of particles can be summed and we can define a differential cross section per unit area/volume as $\sigma_s n_0 p(\hat{s}, \hat{s}')$. Subsequently we can define a scattering, absorption and extinction cross sections per unit area/volume: $\kappa_s = n_0 \sigma_s$, $\kappa_a = n_0 \sigma_a$, $\kappa_e = \kappa_s + \kappa_a = n_0 \sigma_e$.

Power conservation balance leads to the radiative transfer equation [1, 2]:

$$\frac{dI(r, \hat{s})}{ds} = \hat{s} \cdot \nabla_r I(r, \hat{s}) = -\kappa_e I(r, \hat{s}) + \kappa_s \int d\Omega' p(\hat{s}, \hat{s}') I(r, \hat{s}') + \epsilon(r, \hat{s}) \tag{4.1}$$

where $\epsilon$ denotes internal sources.

4.1. Aronson’s theorem

From reciprocity, we show the result in the main text: $\int_{2\pi} G_s(r, \theta'; \theta) d\theta \leq 1$. The equality in the case of negligible absorption is known as Aronson’s theorem [9].

In a general geometrical configuration, one can define a surface Green’s function $G_s(r, \hat{s}; r', \hat{s}')$ [10] giving $I(r, \hat{s})$ for an incident field $I_i = \delta(r_1 - r')\delta(\hat{s}_i - \hat{s}')$ defined on the exterior surface $S$ and with no internal source. And similarly, $G_p(r, \hat{s}; r', \hat{s}')$ [10] the volume Green function with no incident field and a source given by $\epsilon(r, \hat{s}) = \delta(r - r_0)\delta(\hat{s} - \hat{s}_0)$.

If $F(r) = \int I(r, \hat{s}) \hat{s} \, d\hat{s}$ is the flux at the point $r$, then from conservation of energy [1] we have: $\int_S F \cdot \hat{n}_{\text{out}} \, dr = P_e - P_a$, where $P_e$ is the generated power and $P_a$ the absorbed power. For a unit source, $P_e = \int \epsilon(r, \hat{s}) d\hat{s} = 1$, so that:

$$\int_S \int_{\hat{s}_0 \hat{n}_{\text{out}} > 0} G_p(r, \hat{s}; r', \hat{s}') (\hat{s} \cdot \hat{n}_{\text{out}}) d\hat{s} \, d\hat{s}' = P_e - P_a \leq 1 \tag{4.2}$$

Moreover, from reciprocity [10], we have $|\hat{s} \cdot \hat{n}_{\text{out}}| G_p(r, \hat{s}; r_0, \hat{s}_0) = G_s(r_0, -\hat{s}_0; r, -\hat{s})$. This leads after a simple variable change to:

$$\int_S \int_{\hat{s} \cdot \hat{n}_{\text{out}} < 0} G_s(r', \hat{s}'; r, \hat{s}) d\hat{s} \, d\hat{s}' \leq 1 \tag{4.3}$$

4.2. Diffusion equation

Here we reproduce the diffusion equation as in [1, 2] but adjusting the numerical coefficients for a two-dimensional medium.

We first separate the intensity as: $I = I_{ri} + I_d$ where $I_{ri}$ is the reduced (coherent) intensity and $I = I_d$ is the diffuse (incoherent) intensity. The reduced intensity is related to the single scattering and obeys: $\frac{dI_{ri}}{ds} = -\kappa_e I_{ri}$. So from the RTE equations, the diffuse intensity obeys:

$$\frac{dI_d}{ds} = -\kappa_e I_d + \kappa_s \int d\theta' p(\theta, \theta') I_d + J, \quad J = \kappa_s \int d\theta' p(\theta, \theta') I_{ri} \tag{4.4}$$

Now, considering the diffusion approximation, we write: $I_d(r, \theta) = U(r) + \frac{1}{\pi} F(r) \cdot \hat{s}$. This could be seen as a first order series in $\hat{s}$. We also note that the diffuse flux is: $\int I_d \hat{s} \, d\theta = F$.

In order to obtain $U$ and $F$ we apply the operators $\int d\theta$ and $\int d\hat{s} d\theta$ on (4.4). This leads to:

$$\nabla_r \cdot F = -2\pi \kappa_a U + 2\pi \kappa_s U_{ri}, \quad U_{ri}(r) = \frac{1}{2\pi} \int d\theta \, I_{ri}(r, \theta) \tag{4.5}$$

---

\(^1\) If we consider a large bandwidth and/or a large incident angle distribution, interference effects are expected to average out even for a periodic structure as discussed in the main text.
\[ \nabla_r U = -\frac{1}{\pi} \kappa_{tr} F + \frac{1}{\pi} \int d\theta J \hat{s} \]  \hspace{2cm} (4.6) 

where \( \kappa_{tr} = \kappa_e (1 - p_1) \) and \( \kappa_e p_1 = \int d\theta' p(\hat{s}, \hat{s}')[\hat{s} \cdot \hat{s}'] \), so that \( p_1 = \kappa_e \mu / \kappa_e \) where \( \mu \) is the average of the cosine of the scattering angle.

Equations (4.5, 4.6) allow to solve for \( U \) and \( F \). Combining them, we obtain a diffusion equation for \( U \):

\[ \nabla^2 U - \kappa_{tr}^2 U = -2 \kappa_{tr} \kappa_e U_{r1} + \frac{1}{\pi} \nabla \cdot \int d\theta J \hat{s} \]  \hspace{2cm} (4.7) 

Now we need to add appropriate boundary conditions. Supposing that we have a reflection coefficient \( R \) on the surface, this should be: \( I_d(r, \theta) = R(\theta) I_d(r, \pi - \theta) \) for \( \hat{s} \) directed towards the inside of the medium. However, considering the assumed formula for \( I_d \) the condition cannot be satisfied exactly. A common approximate boundary condition is to verify the relation for the fluxes:

\[ \int_{\hat{s} \cdot \hat{n} > 0} I_d(\hat{s} \cdot \hat{n}) d\theta = R(\theta) \int_{\hat{s} \cdot \hat{n} < 0} I_d(\hat{s} \cdot \hat{n}) d\theta \]  \hspace{2cm} (4.8) 

where \( \hat{n} \) is the normal to the surface directed inwards.

Using the formula for \( I_d \) we obtain:

\[ 2(1 - r_1) U + \frac{(1 + r_2)}{2} F \cdot \hat{n} = 0 \]  \hspace{2cm} (4.9) 

4.3. Solution of the diffusion equation for plane-parallel problem

We consider that the medium is infinite along the y-axis so that we have a plane-parallel problem and solve the diffusion equation considering an incident intensity: \( I_{\text{incident}}(\theta_i) = I_0 \delta(\theta_i - \theta) \). Using 4.7 and the boundary condition 4.9, we obtain:

\[ U = C U_{r1} + \frac{I_1}{2\pi} \left[ A e^{-\kappa_e x} + B e^{\kappa_e (x - d)} \right], \quad U_{r1} = \frac{I_1}{2\pi} \left( e^{-\kappa_e x \sec \theta} + \bar{R}_2 e^{\kappa_e (x - 2d) \sec \theta} \right) \]  \hspace{2cm} (4.10) 

where:

\[ I_1 / I_0 = \frac{1 - \bar{R}_1}{1 - \bar{R}_1 \bar{R}_2 Y^2}, \quad \bar{R}_i(\theta), \quad Y = e^{-\kappa_e \sec \theta}, \quad C = \frac{2 - \bar{R}_2 \kappa_e \cos \theta + \bar{R}_1}{\bar{R}_2 Y^2} \]  \hspace{2cm} (4.11) 

\[ \left[ \begin{array}{c} \alpha_1 + \frac{\pi \bar{R}_1}{4Y} e^{-\kappa_e d} \\ \alpha_2 - \frac{\pi \bar{R}_1}{4Y} e^{-\kappa_e d} \end{array} \right] \left[ \begin{array}{c} A \end{array} \right] = \left[ \begin{array}{c} C(1 + \bar{R}_2 Y^2) \alpha_1 + \frac{\pi \bar{R}_1}{4Y} \left( \frac{C}{\cos \theta} + 2p_1 \cos \theta \right) (1 - \bar{R}_2 Y^2) \alpha_2 + \frac{\pi \bar{R}_1}{4Y} \left( \frac{C}{\cos \theta} + 2p_1 \cos \theta \right) (1 - \bar{R}_2) \end{array} \right] \]  \hspace{2cm} (4.12) 

with: \( v_{s,a,e} = \sigma_{s,a,e} n_0 d, \quad v_{tr} = v_e(1 - p_1), \quad v_d^2 = 2 v_{tr} v_a \) and \( \alpha_i = (1 - r_i^2) / (1 + r_2^2) \).

The average interaction factor can be then computed as: \( q = \frac{2\pi}{d} \int_0^d \left( U(x) + U_{r1}(x) \right) / I_0 dx \). So if we use the function \( \xi(x) = (1 - e^{-x}) / x \), then from (4.10) we conclude that:

\[ q = q_0(\theta) \left( \frac{D}{\xi(v_d)} + C \right) + 1 \]  \hspace{2cm} (4.13) 

where:

\[ D = \frac{A + B}{1 + \bar{R}_2 Y}, \quad q_0(\theta) = \frac{(1 - \bar{R}_1)(1 + \bar{R}_2 Y)}{1 - \bar{R}_1 \bar{R}_2 Y^2} \xi(v_e \sec \theta) \]  \hspace{2cm} (4.14)
4.4. Asymmetry factor

The asymmetry factor derived from the previous derivations is \( \mu = \mu_1 \), where in general \( \mu_i = \int_2 \cos(\theta)p(\theta)d\theta \) (where we take \( p(\theta, \theta') = p(\theta - \theta') \)). Since the diffusion result depends only on \( \nu_s, \nu_i \) and \( \mu_1 \), it can be seen as approximating the differential scattering cross section by: \( p(\theta, \theta') = \frac{1}{\pi[1 + 2\mu_1 \cos(\theta - \theta')] \].

The Delta-Eddington approximation [3] allows to incorporate the second moment of \( p \) by including the forward scattering peak using a “delta function” term so that: \( p(\theta, \theta') = \mu_2 \delta(\theta - \theta') + \frac{1 - \mu_2}{\pi[1 + 2\mu \cos(\theta - \theta')]} \), where \( \mu = (\mu_1 - \mu_2)/(1 - \mu_2) \). This approximation matches the Fourier decomposition of \( p \) up to the second term. By incorporating this expression in the RTE (4.1), one recovers a second term with \( p \) replaced by \( \frac{1 - \mu_2}{\pi[1 + 2\mu \cos(\theta - \theta')] \} \) and \( \sigma_s \) replaced by \( \sigma_s/(1 - \mu_2) \). So the diffusion approximation can be made more accurate by replacing \( \mu \) by \( (\mu_1 - \mu_2)/(1 - \mu_2) \) on the spatially resolved scattering peak using a “delta function” term so that: \( p \).

In a three-dimensional medium, \( \mu_i = \int \cos(\theta)p(\theta)d\Omega \) where we take \( \mu_i = \int \cos(\theta)p(\theta)d\Omega \) where in general \( \mu_i = \int_2 \cos(\theta)p(\theta)d\theta \) (where we take \( p(\theta, \theta') = p(\theta - \theta') \)). Since the diffusion result depends only on \( \nu_s, \nu_i \) and \( \mu_1 \), it can be seen as approximating the differential scattering cross section by: \( p(\theta, \theta') = \frac{1}{\pi[1 + 2\mu_1 \cos(\theta - \theta')] \].

5. THREE-DIMENSIONAL MEDIUM

5.1. General result

The coefficients given in section 1 change for a three-dimensional medium. In this case: \( \nu_d^2 = 3\nu_i(\nu_c - \nu_s, \mu) \) and

\[
C = 3[\nu_i(\nu_c + \mu\nu_s)]/[\nu_d^2 - (\nu_c, \nu_i, \sec\theta)^2].
\]

\( D \) is given through boundary conditions by \( D = \frac{A + B}{1 + \hat{R}_2 Y} \), where:

\[
\begin{pmatrix}
\alpha_1 + \frac{\nu_s}{\nu_i} \\
\alpha_2 - \frac{\nu_s}{\nu_i}
\end{pmatrix}
\begin{pmatrix}
A \\
B
\end{pmatrix} = \begin{pmatrix}
C(1 + \hat{R}_2 Y^2)\alpha_1 + \frac{\nu_s}{\nu_i} \left( \frac{C}{\cos^2 \theta} + 3p_1 \cos \theta \right)(1 - \hat{R}_2 Y^2) \\
C(1 + \hat{R}_2)\alpha_2 - \frac{\nu_s}{\nu_i} \left( \frac{C}{\cos^2 \theta} + 3p_1 \cos \theta \right)(1 - \hat{R}_2)Y
\end{pmatrix}
\]

(5.1)

with: \( \alpha_i = (1 - r_1^2)/(1 + r_1^2), r_1^p = \int R(\theta)\cos^2(\theta)d\theta/\int \cos^2(\theta)d\theta, \nu_i p_1 = \nu_s \mu \) and \( \nu_i r_1 = \nu_c(1 - p_1) \).

\[
\eta = \frac{\pi - \int_0^{\pi/2}(q_0^{(1)} + q_0^{(2)})d\theta}{\int_0^{\pi/2} \left[ 0_0^{(1)} D_0^{(1)}(\theta, \nu_i, \nu_i) + 0_0^{(2)} D_0^{(2)}(\theta, \nu_i, \nu_i) \right]/\xi(\nu_c, \sec \theta) - 3 \cos^2 \theta(1 - \hat{R}_2 Y^2)}
\]

(5.2)

with:

\[
(1 + \hat{R}_2 Y)D_0(\theta, \nu_i, \nu_i) = \frac{(\alpha_2 + \frac{\nu_s}{\nu_i})X_1 + (\alpha_1 + \frac{\nu_s}{\nu_i})X_2}{\frac{2}{\nu_i} (\alpha_1 + \alpha_2) + 2\alpha_1 \alpha_2}
\]

(5.3)

where:

\[
X = \begin{pmatrix}
3 \cos^2 \theta(1 + \hat{R}_2 Y^2)\alpha_1 + 2 \cos \theta(1 - \hat{R}_2 Y^2) \\
3 \cos^2 \theta(1 + \hat{R}_2)\alpha_2 - 2 \cos \theta(1 - \hat{R}_2)Y
\end{pmatrix}
\]

(5.4)
5.2. Scattering particles embedded in low-absorbing layer

We consider scattering particles embedded in a layer of index $n$ and negligible absorption in the presence of perfect back-reflector ($R_2 = 1$). In the limit of large scattering we obtain:

$$q = 3 \cos^2 \theta + \frac{2}{\alpha_1} \cos \theta$$  (5.5)

where $\theta$ is the refracted angle ($< \theta_c = \sin \frac{1}{n}$) and $\alpha_1^{-1} = n^2 \left[ 1 + \left( 1 + \frac{1}{n^2} \right)^{1/2} \right]$.

The enhancement with normalized incident flux compared to a single pass is given by: $E(\theta) = q(\theta) / \cos \theta$. For Lambertian source, we have:

$$\frac{\int_{\theta_c}^{\theta} E(\theta) \cos \theta \sin \theta d\theta}{\int_{\theta_c}^{\theta} \cos \theta \sin \theta d\theta} = \frac{2}{1/2n^2} = 4n^2$$  (5.6)
